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Complete differentials of higher order in linear field modules

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Abstract. Complete differentials of higher order in linear field modules are defined. A certain necessary condition for the existence of a complete differential of higher order is given. It is proved that this condition holds in a broad class of linear field modules, which contains differential modules and modules of vector fields on differential spaces of class $\mathcal{D}_0$. Jet-field modules of a linear field module are constructed. The exactness of the sequence of jet-modules is examined. A one-to-one correspondence between complete differentials of higher order and splittings of jet-module sequences is established. An example of a differential space of class $\mathcal{D}_0$ is given in which

1° the module of vector fields over that space in every neighbourhood of a certain point does not possess any vector basis, i.e. it is not differential,

2° a covariant derivative, i.e. a complete differential of the first order, exists in the module of vector fields.

Introduction. We consider a manifold M and a vector bundle ξ over M and we denote (as usual) by $J^k(\xi)$ the vector bundle of holonomic k-order jets of local sections of ξ. The exact sequence of vector bundles

$$0 \to L^k_s(TM, \xi) \to J^k(\xi) \to J^{k-1}(\xi) \to 0$$

called the *sequence of jet-bundles* is well known from the works by R. Palais ([6]), N. V. Que ([9]), D. Spencer ([13], [14]) and others. A differential operator of order k, corresponding to a splitting of the sequence, is termed by Palais a complete differential of order k in a bundle ξ. In the case $k = 1$ it is simply a covariant derivative.

In the present work we consider an arbitrary differential space $(M, \mathcal{C})$ ([5], [10]) instead of a manifold M and an arbitrary linear field module $\mathcal{W}$ instead of the module of sections of a vector bundle ξ. Differential spaces have been examined in the works by R. Sikorski ([11], [12]), W. Waliszewski ([18], [19]) as well as in the works by P. Walczak ([15]–[17]), K. Cegiełka ([2], [3]), M. Pustelnik ([8]) and others. Linear field modules defined on differential spaces were introduced by R. Sikorski ([11]).

In the present work we shall construct a linear field module $J^k(\mathcal{W})^{\sim}$ and an exact sequence of jet-modules analogous to the sequence of jet-bundles and we shall prove the equivalence between the definition of a complete differential as a certain differential operator of order k and as a splitting of the jet-module sequence. The construction of the module $J^k(\mathcal{W})$ will be possible under certain assumptions about the module $\mathcal{W}$; the exactness of the jet-module sequence will occur in certain conditions. These assumptions and conditions will be examined more precisely for a class of pseudo-differential modules, which contains differential modules ([11], [12]) and modules of vector fields on a differential space of the class $\mathcal{D}_0$ ([16], [17]). K. Cegiełka in [2] showed that if a linear field module $\mathcal{W}$ on a differential space $(M, \mathcal{C})$ is differential and if it is possible to subordinate a smooth partition of unity to every open covering of the space $(M, \tau_{\mathcal{C}})$, then there exists in $\mathcal{W}$ a scalar product, and so a covariant derivative also exists. It turns out that the existence of a scalar product does not imply the existence of a local basis in the module under consideration. An adequate example will be given at the end of section 3.

1. Preliminaries. Differential spaces discussed in this paper as well as the notions of a tangent vector, tangent space, smooth mapping, tangent mapping, smooth vector field and the denotations $\tau_{\mathcal{C}}$ and $\mathcal{C}_A$ have been adopted from the works by R. Sikorski [10], [12]. A $\mathcal{C}$-module of smooth vector fields on a differential space $(M, \mathcal{C})$ will be denoted by $\mathcal{X}(M, \mathcal{C})$ and the vector subspace of the tangent space $(M, \mathcal{C})_p$, $p \in M$, consisting of these vectors which are values of a smooth vector field will be denoted by $(M, \mathcal{C})'_p$.

In a differential space $(M, \mathcal{C})$ whose topology is paracompact and locally compact, for any open covering there exists a smooth partition of unity subordinated to this covering; this fact has been proved by K. Cegiełka ([2]), M. Pustelnik in [8] proved that the assumption of local compactness may be replaced by $\mathcal{C}$-normality. It is easy to show that the assumptions of $\mathcal{C}$-normality is weaker than that of local compactness (assuming paracompactness) and equivalent to the existence of a smooth partition of unity subordinate to an arbitrary open covering.

1.1. Differential spaces of class $\mathcal{D}_0$. The existence and specification of the widest class of differential spaces in which the theorem on a diffeomorphism holds was a problem raised by Waliszewski and solved by Walczak in his paper [16]. Paper [17] was devoted to the investigation of that class.

THEOREM 1.1.1. *If $(M, \mathcal{C})$ is a differential space of class $\mathcal{D}_0$, then the set M' of all points $p \in M$ for which*

$$(M, \mathcal{C})_p = (M, \mathcal{C})'_p$$

is open and dense in topology $\tau_{\mathcal{C}}$.

Proof. The openness of M' is evident from the definition of this set. For a non-negative integer n, let M_n be the set of all points $p \in M$ for which $\dim(M, \mathscr{C})_p = n$. It is easy to see that

$$M' = \bigcup_n \operatorname{Int} M_n.$$

To prove that $\overline{M'} = M$ we shall show that every point $p \in M$ has a neighbourhood $U \in \tau_{\mathscr{C}}$ such that

$$U \subset \overline{M' \cap U}. \tag{1.1.1}$$

We take a set U covering p such that $\dim(M, \mathscr{C})_q \leqslant \dim(M, \mathscr{C})_p$ for $q \in U$. Obviously, if $n = \dim(M, \mathscr{C})_p$, then

$$M' \cap U = \bigcup_{k=0}^{n} ((\operatorname{Int} M_k) \cap U) = \bigcup_{k=0}^{n} \operatorname{Int}(M_k \cap U).$$

Let $A_k = (M_k \cap U) \setminus \operatorname{Int}(M_k \cap U)$, $k = 0, 1, \ldots, n$. Since

$$U \setminus (M' \cap U) = \bigcup_{k=0}^{n} A_k,$$

to show inclusion (1.1.1) it suffices to prove the equality

$$\operatorname{Int}\Big(\bigcup_{k=0}^{r} A_k\Big) = \emptyset, \quad r = 0, 1, \ldots, n. \tag{1.1.2}$$

We apply induction on r. Since $M_0 \cap U$ is open, equality (1.1.2) is satisfied for $r = 0$. Assume that (1.1.2) is satisfied for an integer $r < n$. From the openness of the set $U \cap (M_0 \cup \ldots \cup M_r)$ and the equality $\overline{A_{r+1}} \cap \cap (M_0 \cup \ldots \cup M_r) \cap U = \emptyset$ results

$$\operatorname{Int}\Big(\bigcup_{k=0}^{r+1} A_k\Big) = \Big(\operatorname{Int} \bigcup_{k=0}^{r} A_k\Big) \cup \operatorname{Int} A_{r+1} = \emptyset. \quad \text{q.e.d.}$$

The above theorem states that, in general, there are "many" vector fields in a differential space of class $\mathscr{D}_0$.

1.2. Examples of differential spaces.

1.2.1. Let M and N be manifolds of class C^∞ and let $f\colon M \to N$; denote by $\mathscr{T}(M)$ and $\mathscr{T}(N)$ the rings of smooth functions on M and N; then the differential spaces $(f[M], \mathscr{T}(N)_{f[M]})$ and $(f^{-1}[\{a\}], \mathscr{T}(M)_{f^{-1}[\{a\}]})$, where $a \in N$ are not in general submanifolds.

1.2.2. The differential space $(M \times N, \mathscr{T}(M) \times \mathscr{T}(N))$ ([13]) is not a manifold if M and N are manifolds with a boundary.

1.2.3. Let N, N' be submanifolds of M. The differential spaces $(N \cap N', \mathscr{T}(M)_{N \cap N'})$ and $(N \cup N', \mathscr{T}(M)_{N \cup N'})$ need not be submanifolds.

1.2.4. On a manifold M, an arbitrary collection of vector fields

$X_1, \ldots, X_k$ defines several subspaces of the space $(M, \mathcal{T}(M))$ of the form $(A, \mathcal{T}(M)_A)$, where, for example,

(a) $A = \{p \in M;\ X_1(p) = \ldots = X_k(p) = 0\}$,

(b) $A = \{p \in M;$ the vectors $X_1(p), \ldots, X_k(p)$ are lineary independent$\}$.

1.2.5. Let K be a solid in $\boldsymbol{R}^n$. A differential space $(K, C^\infty(\boldsymbol{R}^n)_K)$ and the k-dimensional skeletons of this solid with the differential structure induced from $\boldsymbol{R}^n$ need not be manifolds. However, the solid may be a union (in the sense of example 3) of manifolds with a boundary.

1.2.6. Let (M, g) be a Riemannian manifold. Let us fix point $p \in M$ and denote by $C(p)$ the set of vectors $v \in M_p$ for which the differential $(d\ \mathrm{Exp}_p)_v$ is not an isomorphism. The corresponding differential subspaces $C(p)$ and $\mathrm{Exp}_p[C(p)]$ of the spaces M_p and M need not be submanifolds.

1.2.7. We define a structure $\mathscr{C}$ on the set R of real numbers by the formula

$$\mathscr{C} = (S_C C_0)_{\boldsymbol{R}}, \quad \text{where } C_0 = \{R \ni t \mapsto |t-s| \in R;\ s \in R\}.$$

Then $\dim(R, C)_t = 2$ and $\dim(R, C)'_t = 0$ for any point $t \in R$.

The spaces in examples 1–6 are obviously of class $\mathscr{D}_0$, while in the last example the space $(R, \mathscr{C})$ is not of class $\mathscr{D}_0$, according to Theorem 1.1.1.

1.3. Linear field modules.

Definition 1.3.1. A *linear field module* is a triple $\mathscr{W} = ((M, \mathscr{C}), \Phi, \mathscr{W})$, where $(M, \mathscr{C})$ is a differential space, Φ is a function assigning vector spaces $\Phi(p)$ to points $p \in M$ and $\mathscr{W}$ is a certain $\mathscr{C}$-module of linear Φ-fields satisfying the condition:

If W is a linear Φ-field such that for any point $p \in M$ there exist a neighbourhood $U \in \tau_{\mathscr{C}}$ of this point and a field $V \in \mathscr{W}$ such that $W|U = V|U$, then $W \in \mathscr{W}$.

A module $\mathscr{W}$ satisfying the last condition is said to be closed with respect to localization.

We shall denote by $\Phi_{\mathscr{W}}(p)$ the vector space consisting of vectors $v \in \Phi(p)$ which are the values of fields from the module $\mathscr{W}$.

Suppose that with every point $p \in M$ there is associated a linear mapping $L(p)\colon \Phi_{\mathscr{W}}(p) \to \Psi_{\mathscr{V}}(p)$ satisfying the condition

$$L(W) = \big(M \ni p \mapsto L(p)(W(p))\big) \in \mathscr{V} \quad \text{for } W \in \mathscr{W};$$

then L is called a *homomorphism of the linear field module* $((M, \mathscr{C}), \Phi, \mathscr{W})$ *into the linear field module* $((M, \mathscr{C}), \Psi, \mathscr{V})$. Then $L\colon \mathscr{W} \to \mathscr{V}$ is a homomorphism of $\mathscr{C}$-modules.

A homomorphism of $\mathscr{C}$-modules $L\colon \mathscr{W} \to \mathscr{V}$ induces a homomorphism of linear field modules if and only if it satisfies the following condition:

if $W \in \mathscr{W}$ and $W(p) = 0$, then $L(W)(p) = 0$;

if $\mathscr{V}$ and $\mathscr{W}$ are modules of Φ and θ-linear fields on a differential space $(M, \mathscr{C})$, then we denote by $L^k_s(\mathscr{V}, \mathscr{W})$ the module of all linear Ψ-fields L, where $\Psi(p) = L^k_s(\Phi_{\mathscr{V}}(p); \theta_{\mathscr{W}}(p))$, $p \in M$, such that $L(V_1, \ldots, V_k) \in \mathscr{W}$ for $V_1, \ldots, V_k \in \mathscr{V}$. The module $L(\mathscr{V}, \mathscr{C})$ will be denoted by $\mathscr{V}^*$.

An example of a differential space $(M, \mathscr{C})$, a linear field module $\mathscr{W}$ and a $\mathscr{C}$-linear mapping from $\mathscr{X}(M, \mathscr{C})$ into $\mathscr{W}$ which is not a linear Ψ-field will be given at the end of section 3. However, if every vector field $V \in \mathscr{X}(M, \mathscr{C})$ equal 0 at p is of the form $V = \sum_{i=1}^{n} f^i W_i$ for some functions $f^i \in \mathscr{C}$ such that $f^i(p) = 0$ and fields $W_i \in \mathscr{X}(M, \mathscr{C})$, $i = 1, \ldots, n$, then every $\mathscr{C}$-multilinear mapping from the module $\mathscr{X}(M, \mathscr{C})$ into $\mathscr{W}$ is a linear Ψ-field.

1.4. Pseudo-differential modules.

Definition 1.4.1. A linear field module $((M, \mathscr{C}), \Phi, \mathscr{W})$ is called a *pseudo-differential module* if for any point $q \in M$ there exist a neighbourhood $U \in \tau_{\mathscr{C}}$ of this point and a differential module $((U, \mathscr{C}_U), \Psi, \mathscr{V})$ such that $\Phi(p) \subset \Psi(p)$ for $p \in U$ and

(1.4.1) if $V \in \mathscr{V}$ and $V(p) \in \Phi_{\mathscr{W}}(p)$ for any point $p \in U$, then $V \in \mathscr{W}_U$.

Differential modules and modules of smooth vector fields on a differential space of class $\mathscr{D}_0$ are examples of pseudo-differential modules. Basic properties of pseudo-differential modules are given underneath:

Theorem 1.4.1. *If $((M, \mathscr{C}), \Phi, \mathscr{W})$ is a pseudo-differential module, then*:

(1) $\Psi_{\mathscr{W}^*}(p) = (\Phi_{\mathscr{W}}(p))^*$, *where* $\Psi(p) = (\Phi_{\mathscr{W}}(p))^*$, $p \in M$; *i.e. for any linear mapping* $\tau\colon \Phi_{\mathscr{W}}(p) \to R$ *there exists a field* $h \in \mathscr{W}^*$ *such that* $h(p) = \tau$;

(2) *if W is a $\Phi_{\mathscr{W}}$-linear field such that for any field $h \in \mathscr{W}^*$ the function $h \circ W$ belongs to the ring $\mathscr{C}$, then $W \in \mathscr{W}$*;

(3) *this module is reflexive, i.e. the mapping $H_{\mathscr{W}}\colon \mathscr{W} \to \mathscr{W}^{**}$ defined by the formula $H_{\mathscr{W}}(W) = (\mathscr{W}^* \ni h \mapsto h \circ W \in \mathscr{C})$, $W \in \mathscr{W}$, is a linear field module isomorphism.*

Remark. Actually, it will be proved that the relations $1 \Rightarrow (2 \Leftrightarrow 3)$ hold for any linear field modules.

(a) $(1 \wedge 2) \Rightarrow 3$. It suffices to prove that $\ker H_{\mathscr{W}} = 0$ and $\operatorname{im} H_{\mathscr{W}} = \mathscr{W}^{**}$. If $H_{\mathscr{W}}(W) = 0$ for a certain field $W \in \mathscr{W}$, then $h(p)(W(p)) = 0$ for every field $h \in \mathscr{W}^*$. From condition (1) follows the equality $W(p) = 0$. Now consider a field $L \in \mathscr{W}^{**}$. From assumption (1) it follows that for any point $p \in M$ there is exactly one element $v \in \Phi_{\mathscr{W}}(p)$ such that $L(p)(\tau) = \tau(v)$ for $\tau \in \Psi_{\mathscr{W}^*}(p)$. This defines a certain linear $\Phi_{\mathscr{W}}$-field W for which $h \circ W = (M \ni p \mapsto h(p)(W(p))) = (M \ni p \mapsto L(p)(h(p))) = L(h) \in \mathscr{C}$ for every field $h \in \mathscr{W}^*$. From the assumption (2) it follows that $W \in \mathscr{W}$.

(b) $(1 \wedge 3) \Rightarrow 2$. If W is an arbitrary linear $\Phi_{\mathscr{W}}$-field such that $h \circ W \in \mathscr{C}$ for any field $h \in \mathscr{W}^*$, then $(\mathscr{W}^* \ni h \mapsto h \circ W \in \mathscr{C}) \in \mathscr{W}^{**}$. Hence there exists exactly one field $W' \in \mathscr{W}$ such that $h \circ W = h \circ W'$ for any field $h \in \mathscr{W}^*$. In view of condition (1) we have the equality $W = W'$.

Proof of the theorem. Obviously, it suffices to check that a pseudo-differential module fulfils conditions (1) and (2). Let us take a point $q \in M$, a neighbourhood $U \in \tau_{\mathscr{C}}$ of q and a differential module $((U, \mathscr{C}_U), \theta, \mathscr{V})$ such that $\Phi(p) \subset \theta(p)$ for $p \in U$ and condition (1.4.2) is fulfilled.

(1) Let $\tau\colon \Phi_{\mathscr{W}}(q) \to R$ be an arbitrary linear mapping and let $\varrho\colon \theta(q) \to R$ be a certain linear extension of it. Let us take an arbitrary field $F \in \mathscr{V}^*$ such that $F(q) = \varrho$. Obviously, the field $F' = F|\Phi_{\mathscr{W}}$ defined by the formula $(F|\Phi_{\mathscr{W}})(p) = F(p)|\Phi_{\mathscr{W}}(p)$, $p \in U$, is an element of the module $(\mathscr{W}_U)^*$ and has the property: $F'(q) = \tau$. Taking into account the $\mathscr{C}$-regularity of the space $(M, \tau_{\mathscr{C}})$ ([14]), we see that condition (1) is fulfilled.

(2) Let W be an arbitrary linear $\Phi_{\mathscr{W}}$-field such that $h \circ W \in \mathscr{C}$ for any field $h \in \mathscr{W}^*$. In particular, $F \circ (W|U) = F|\Phi_{\mathscr{W}} \circ (W|U) \in \mathscr{C}$ for any field $F \in \mathscr{V}^*$. Therefore $W|U \in \mathscr{V}$, and further, from assumption (1.4.1), it follows that $W|U \in \mathscr{W}_U$. Hence $W \in \mathscr{W}$. q.e.d.

1.5. Examples of linear field modules.

1.5.1. Let ξ and η be vector bundles over manifolds M and N, respectively, and let $\alpha\colon \xi \to \eta$ be a morphism of vector bundles, i.e. a smooth mapping such that $\alpha_p = \alpha|\xi_p\colon \xi_p \to \eta_{f(p)}$, $p \in M$ is a linear mapping, where $f\colon M \to N$. Let $\mathscr{W}$ be a submodule of the module $C^\infty(\xi)$ consisting of sections σ for which $\sigma(p) \in \ker \alpha_p$, $p \in M$, and $\mathscr{V}$ a submodule of $C^\infty(f^*\eta)$ consisting of fields σ for which $\sigma(p) \in \alpha_p[\xi_p]$, $p \in M$. The linear field modules

$$(M, (M \ni p \mapsto \ker \alpha_p), \mathscr{W}), \quad (M, (M \ni p \mapsto \operatorname{im} \alpha_p), \mathscr{V})$$

are not, in general, differential modules (i.e. $\bigcup_{p \in M} \ker \alpha_p$ and $\bigcup_{p \in M} \operatorname{im} \alpha_p$ generally are not subbundles of ξ and $f^*\eta$, respectively).

1.5.2. Let ξ and η be vector bundles over a manifold M and Φ a differential operator of order k from the bundle ξ into η. Following Spencer ([13], [14]), we denote by φ the corresponding morphism of the vector bundle $J^k(\xi)$ into η, by $P_l(\varphi)$ its l-th extension $P_l(\varphi)\colon J^{k+l}(\xi) \to J^l(\eta)$ and by $\sigma_l(\varphi)$ the unique linear morphism $\sigma_l(\varphi)\colon S^{k+l}T^* \otimes \xi \to S^l T^* \otimes \xi$, $l \geqslant 0$, such that the following diagram is commutative:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
S^{k+l}T^* \otimes \xi & \xrightarrow{\sigma_l(\varphi)} & S^l T^* \otimes \xi \\
\downarrow & & \downarrow \\
J^{k+l}(\xi) & \xrightarrow{P_l(\varphi)} & J^l(\xi) \\
\downarrow & & \downarrow \\
J^{k+l-1}(\xi) & \xrightarrow{P_{l-1}(\varphi)} & J^{l-1}(\xi) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Let $g_{k+l} = \ker \sigma_l(\varphi) \subset S^{k+l} T^* \otimes \xi$ and $R_{k+l} = \ker P_l(\varphi) \subset J^{k+l}(\xi)$. Adequate linear field modules (constructed accordingly to the scheme from the former example) with values in g_{k+l} and R_{k+l}, respectively, are not differential, in general.

1.5.3. Let M be a manifold. An arbitrary collection of smooth vector fields $X_1, \ldots, X_k$ defines a linear field module in which $\Phi(p)$, $p \in M$, is the vector space spanned by the vectors $X_1(p), \ldots, X_k(p)$.

1.5.4. Let us consider a curve $f\colon (a, b) \to \boldsymbol{R}^n$ of class C^∞ and at every point $p \in (a, b)$ the osculating space of order k to f in the sense of E. Cartan ([1]), i.e. the plane in $\boldsymbol{R}^n$ spanned by the points: $f(p)$, $f(p)+f'(p)$, $f(p)+f''(p), \ldots, f(p)+f^{(k)}(p)$. Let us produce a linear field module in which $\Phi(p)$, $p \in (a, b)$, will be the osculating space of order k to the curve f and all mappings $V\colon (a, b) \to \boldsymbol{R}^n$ of class C^∞, such that $V(p) \in \Phi(p)$, $p \in (a, b)$, will form a linear Φ-field module. The generated linear field module need not be differential. A generalization of the above definition of the osculating space to a curve in the case of a realization f of a manifold M in the space $\boldsymbol{R}^n, f\colon M \to \boldsymbol{R}^n$, was given by W. Pohl ([7]). Proceeding as above, we can again define a linear field module which, in general is not differential.

2. Ideals $I_p^{(k)}(M, \mathscr{C})$.

DEFINITION 2.1. For an arbitrary differential space $(M, \mathscr{C})$ and a point $p \in M$ we define by induction the sets $I_p^{(k)}(M, \mathscr{C})$, $k \in N$, in the following way:

(a) $I_p^{(1)}(M, \mathscr{C}) = I_p(M, \mathscr{C})$ equals the set of functions $f \in \mathscr{C}$ for which $f(p) = 0$;

(b) $f \in I_p^{(k+1)}(M, \mathscr{C})$ if and only if $f \in I_p^{(k)}(M, \mathscr{C})$ and for any collection of vector fields $X_1, \ldots, X_k \in \mathscr{X}(M, \mathscr{C})$ the equality

$$[(X_1, \ldots, X_k) f](p) = 0$$

holds.

Note that:

(2.1) The sets $I_p^{(k)}(M, \mathscr{C})$, $k \in N$, are ideals in the ring $\mathscr{C}$,

(2.2) If $f \in I_p^{(k+1)}(M, \mathscr{C})$, $1 \leqslant r \leqslant k$, $X_1, \ldots, X_r \in \mathscr{X}(M, \mathscr{C})$, then $(X_1, \ldots, X_r) f \in I_p^{(k+1-r)}(M, \mathscr{C})$,

(2.3) $[(I_p^{(1)}(M, \mathscr{C}))^k]_M \subset I_p^{(k)}(M, \mathscr{C})$.

As a rule, inclusion (2.3) cannot be replaced by an equality.

EXAMPLE 2.1. Let $A \subset R^2$ be the set of points (x, y) for which $x = 0$ or $y = 0$ and let $D = C^\infty(R^2)_A$. Obviously, the dimension of the space $(A, D)_{(0,0)}$ is equal 2; moreover, since every smooth vector field on (A, D) is equal 0 at the point $(0, 0)$, the dimension of the space $(A, D)'_{(0,0)}$ is equal 0. Consequently $I_p^{(k)}(A, D) = I_p^{(1)}(A, D)$ for $k \geqslant 1$. There exists a function

$\alpha \in I_p^{(1)}(A, D)$ such that $(d\alpha)_{(0,0)} \neq 0$; so $\alpha \notin ((I_p^{(1)}(A, D))^2)_A$ and also $\alpha \notin ((I_p^{(1)}(A, D))^k)_A$.

THEOREM 2.1. *For any differential space* $(M, \mathscr{C})$, *any point* $p \in M$ *and any positive integer* k *there exists exactly one linear mapping*

$$d_p^{(k)}: I_p^{(k)}(M, \mathscr{C}) \to L_s^k((M, \mathscr{C})_p', R)$$

such that for vector fields $X_1, \ldots, X_k \in \mathscr{X}(M, \mathscr{C})$ *and functions* $f \in I_p^{(k)}(M, \mathscr{C})$ *the equality*

$$(d_p^{(k)} f)(X_1(p), \ldots, X_k(p)) = [(X_1, \ldots, X_k) f](p)$$

holds. Moreover, the sequence

$$(2.4) \qquad 0 \to I_p^{(k+1)}(M, \mathscr{C}) \hookrightarrow I_p^{(k)}(M, \mathscr{C}) \xrightarrow{d_p^{(k)}} L_s^k((M, \mathscr{C})_p', R) \to 0$$

is exact if $\dim(M, \mathscr{C})_p' < \infty$.

Proof. The existence of the mapping $d_p^{(k)}$, its uniqueness and linearity may be checked just as in the case when $(M, \mathscr{C})$ is a manifold ([6]). To prove the exactness of the sequence (2.4) it suffices to show the surjectivity of the mapping $d_p^{(k)}$ in the case when $\dim(M, \mathscr{C})_p' > 0$. Let $\alpha: \otimes^k((M, \mathscr{C})_p')^* \to L^k((M, \mathscr{C})_p', R)$ be the natural linear isomorphism. Let us fix a basis $v_1, \ldots, v_n$ of the space $(M, \mathscr{C})_p'$ and take arbitrary vector fields $X_1, \ldots, X_k \in \mathscr{X}(M, \mathscr{C})$ such that $X_i(p) = v_i$, $i = 1, \ldots, n$. There exist functions $\beta_1, \ldots, \beta_n \in \mathscr{C}$ such that $\beta_j(p) = 0$ and $X_i(\beta_j) = \delta_{ij}$, $i, j \leqslant n$ ([12]). An arbitrary element τ of the space $\otimes^k((M, \mathscr{C})_p')^*$ is of the form

$$\tau = \sum_{i_1, \ldots, i_k = 1} a_{i_1, \ldots, i_k} d_p^{(1)} \beta_{i_1} \otimes \ldots \otimes d_p^{(1)} \beta_{i_k}$$

with uniquely determined numbers $a_{i_1, \ldots, i_k} \in R$. $\alpha(\tau)$ is a symmetric mapping if and only if the matrix

$$[a_{i_1, \ldots, i_k};\ 1 \leqslant i_1, \ldots, i_k \leqslant n]$$

is symmetric.

Let now $\alpha(\tau)$ be an arbitrary element of the space $L_s^k((M, \mathscr{C})_p', R)$. Let

$$f = \sum_{\substack{\alpha_1 + \ldots + \alpha_n = k \\ 0 \leqslant \alpha_1, \ldots, \alpha_n \leqslant k}} \frac{1}{\alpha_1! \cdot \ldots \cdot \alpha_n!} \beta_1^{\alpha_1} \cdot \ldots \cdot \beta_n^{\alpha_n} a_{(\alpha_1, \ldots, \alpha_n)},$$

where the number $a_{(\alpha_1, \ldots, \alpha_n)}$ is equal $a_{i_1, \ldots, i_k}$ for the sequence $i_1, \ldots, i_k$ constructed in the following way: at the beginning the number 1 appears α_1 times, then the number 2 is repeated α_2 times etc., the number n occurs α_n

times. It is clear that for a sequence $\alpha'_1, \ldots, \alpha'_n$ such that $\alpha'_1 + \ldots + \alpha'_n = k$ and $0 \leqslant \alpha'_1, \ldots, \alpha'_n \leqslant k$

$$(d_p^{(k)} f)\big(\underbrace{X_1(p), \ldots, X_1(p)}_{\alpha'_1 \text{ times}}, \ldots, \underbrace{X_n(p), \ldots, X_n(p)}_{\alpha'_n \text{ times}}\big)$$

$$= (d_p^{(k)} f)\big(X_1^{\alpha'_1}(p), \ldots, X_n^{\alpha'_n}(p)\big)$$

$$= \sum_{\substack{\alpha_1 + \ldots + \alpha_n = k \\ \alpha_1, \ldots, \alpha_n \geqslant 0}} \frac{1}{\alpha_1! \cdot \ldots \cdot \alpha_n!} a_{(\alpha_1, \ldots, \alpha_n)} (X_1^{\alpha'_1}, \ldots, X_n^{\alpha'_n})(\beta_1^{\alpha_1}, \ldots, \beta_n^{\alpha_n})(p)$$

$$= \sum_{\substack{\alpha_1 + \ldots + \alpha_n = k \\ \alpha_1, \ldots, \alpha_n \geqslant 0}} \frac{1}{\alpha_1! \cdot \ldots \cdot \alpha_n!} a_{(\alpha_1, \ldots, \alpha_n)} \alpha_1! \cdot \ldots \cdot \alpha_n! \, \delta_{\alpha_1}^{\alpha'_1} \cdot \ldots \cdot \delta_{\alpha_n}^{\alpha'_n}$$

$$= a_{(\alpha'_1, \ldots, \alpha'_n)} = \alpha(\tau)\big(\underbrace{X_1(p), \ldots, X_1(p)}_{\alpha'_1 \text{ times}}, \ldots, \underbrace{X_n(p), \ldots, X_n(p)}_{\alpha'_n \text{ times}}\big)$$

q.e.d.

There exists a differential space $(M, \mathscr{C})$ and a point $p \in M$ at which $\dim(M, \mathscr{C})_p = \infty$ and $\dim(M, \mathscr{C})'_p < \infty$.

EXAMPLE 2.2. Let (A, D) be a differential space from Example 2.1. Let us take $(M, \mathscr{C}) = \bigtimes_{m \in N} (A_m, D_m)$, where $(A_m, D_m) = (A, D)$, $m = 1, 2, \ldots$, and a point $p \in M$ such that $pr_n(p) = (0, 0)$. It can be proved that $\dim(M, \mathscr{C})_p = \infty$ and $\dim(M, \mathscr{C})'_p = 0$.

LEMMA 2.1. *If functions $f^1, \ldots, f^n$ belong to the ideal $I_p^{(k)}(M, \mathscr{C})$ and $g_1, \ldots, g_n$ are arbitrary functions of class $\mathscr{C}$, then*

$$d_p^{(k)}\Big(\sum_{i=1}^{n} f^i g_i\Big) = \sum_{i=1}^{n} (d_p^{(k)} f^i) g_i(p).$$

Proof. The proof will be inductive on k. By the linearity of $d_p^{(k)}$ it suffices to prove the equality for $n = 1$. Let $f \in I_p^{(k)}(M, \mathscr{C})$ and $g \in \mathscr{C}$. When $k = 1$ the proof is evident. Let $k > 1$,

$$d_p^{(k)}(f \cdot g)(X_1(p), \ldots, X_k(p)) = [(X_1, \ldots, X_k)(f \cdot g)](p)$$
$$= [(X_1, \ldots, X_{k-1})(X_k(f \cdot g))](p)$$
$$= [(X_1, \ldots, X_{k-1})((X_k f) g + f(X_k g))](p)$$
$$= [(X_1, \ldots, X_{k-1})((X_k f) g)](p) + [(X_1, \ldots, X_{k-1})(f(X_k g))](p)$$
$$= d_p^{(k-1)}((X_k f) g)(X_1(p), \ldots, X_{k-1}(p)) + 0$$
$$= [d_p^{(k-1)}(X_k f) g(p)](X_1(p), \ldots, X_{k-1}(p))$$
$$= d_p^{(k-1)}(X_k f)(X_1(p), \ldots, X_{k-1}(p)) \cdot g(p)$$

$$= [(X_1, \ldots, X_{k-1})(X_k f)](p)\cdot g(p)$$
$$= [(X_1, \ldots, X_k) f](p)\cdot g(p)$$
$$= [(d_p^{(k)} f)\cdot g(p)]\big(X_1(p), \ldots, X_k(p)\big). \quad \text{q.e.d.}$$

3. Modules of jets. An exact sequence of jet-modules.

3.1. Opening remarks. For an arbitrary linear field module $\mathscr{W} = ((M, \mathscr{C}), \Phi, \mathscr{W})$ we shall look for the possibly weakest conditions under which a linear field module $J^k(\mathscr{W})$, called the *module of jets of order k* of the module $\mathscr{W}$, can be rationally defined.

The notion of jet appeared in Ch. Ehresmann's work [4]. In the same series of articles we can find also the notion of a holonomic extension of order k of a bundle ξ. In the case of linear bundles this notion was introduced in a way different but equivalent and more useful for us by R. Palais ([16]) in the course of presenting the theory of differential operators.

The definition of the jet field module $J^k(\mathscr{W})$ in the case of a linear field module is a generalization of this construction.

3.2. Definition of a complete differential of higher order in a linear field module. Examples.

Definition 3.2.1. A *complete differential of order k in a linear Φ-field module* $\mathscr{W}$ *over a differential space* $(M, \mathscr{C})$ is defined as an R-linear mapping

$$D^k\colon \mathscr{W} \to L_s^k(\mathscr{X}(M, \mathscr{C}), \mathscr{W})$$

satisfying the condition

$$(3.2.1) \qquad \big(D^k(f\cdot W)\big)(p) = d_p^{(k)}\big(f - f(p)\big)\otimes W(p) + f(p)(D^k W)(p)$$

for fields $W\in\mathscr{W}$, points $p\in M$ and functions $f\in\mathscr{C}$ such that

$$f - f(p)\in I_p^{(k)}(M, \mathscr{C}).$$

For $k = 1$ we have the ordinary definition of a covariant derivative. We shall further denote $(D^k W)(p)$ by $D_p^k(W)$.

Example 3.2.1. A fundamental example of a complete differential of order k is the mapping

$$d^k\colon C^\infty(\mathbf{R}^n) \to L_s^k\big(\mathscr{X}(\mathbf{R}^n, C^\infty(\mathbf{R}^n)), C^\infty(\mathbf{R}^n)\big)$$

define by the formula

$$(d^k f)(X_1, \ldots, X_k)(p) = \sum_{i_1,\ldots,i_k=1} X_1(p)(pr_{i_1})\cdot\ldots\cdot X_k(p)(pr_{i_k}) f_{|i_1\ldots i_k}(p)$$

for $X_1, \ldots, X_k$ smooth vector fields on $\mathbf{R}^n$ and $pr_j\colon \mathbf{R}^n\to\mathbf{R}$, $j = 1, \ldots, n$, the natural projections.

Thus, in order to evaluate $(d^k f)(X_1, \ldots, X_k)(p)$, the vectors

$X_2(p), \ldots, X_k(p)$ should be extended to vector fields $Y_2, \ldots, Y_k$, constant with respect to the natural covariant derivative in the module $\mathscr{X}(\boldsymbol{R}^n, C^\infty(\boldsymbol{R}^n))$ and the following quantity should be computed:

$$(d^k f)(X_1, \ldots, X_k)(p) = X_1(p)[(Y_2, \ldots, Y_k) f].$$

EXAMPLE 3.2.2. Let us consider a vector bundle ξ over a manifold M, a covariant derivative $\tilde{\nabla}$ in the tangent bundle TM with vanishing curvature tensor and a covariant derivative ∇ in ξ such that, whenever $\bar{X}$ and $\bar{Y}$ are $\tilde{\nabla}$-constant fields defined on an open set $U \subset M$, the curvature tensor of ∇ satisfies

$$R_{\bar{X},\bar{Y}}\sigma = -\nabla_{[\bar{X},\bar{Y}]}\sigma,$$

σ being any section of ξ over U. For vector field X on the manifold M and a point $p \in M$ we denote by $\bar{X}^p$ the $\tilde{\nabla}$-constant field defined in a certain neighbourhood of p such that $X(p) = \bar{X}^p(p)$. Let

$$(D_{X_1,\ldots,X_k}\sigma)(p) = \big(\nabla_{\bar{X}_1^p}(\ldots(\nabla_{\bar{X}_k^p}\sigma)\ldots)\big)(p).$$

The operator D defined in this way is a complete differential of order k.

3.3. The modules $Z_p^{(k)}(\mathscr{W})$ and $Z_p^k(\mathscr{W})$. Let us consider a certain vector bundle ξ over a manifold M. R. Palais [6] has defined, for an arbitrary point $p \in M$ and an integer $k \geqslant 0$, a submodule $Z_p^k(\xi)$ of $C^\infty(\xi)$ (the module of global sections of ξ) to be equal $I_p^k(M)C^\infty(\xi)$. It corresponds to these global sections whose holonomic k-jet at p (in the terminology of Ehresmann) is equal to 0. If D^k is a complete differential of order k in the module $C^\infty(\xi)$, then $Z_p^k(\xi)$ consists of these sections $\sigma \in Z_p^{k-1}(\xi)$ for which $D_p^k(\sigma) = 0$.

DEFINITION 3.3.1. Assume, for an arbitrary linear field module $\mathscr{W} = ((M, \mathscr{C}), \Phi, \mathscr{W})$ and a point $p \in M$, that

(a) $Z_p^{(k)}(\mathscr{W}) = I_p^{(k+1)}(M, \mathscr{C})\mathscr{W}$, $k = 0, 1, 2, \ldots$

(b) $Z_p^0(\mathscr{W}) = Z_p^{(0)}(\mathscr{W})$ and $Z_p^k(\mathscr{W})$, $k = 1, 2, \ldots$, is equal to the submodule of $\mathscr{W}$ containing these and only these fields $W \in Z_p^{(k-1)}(\mathscr{W})$ which can be written in the form $W = \sum_{i=1}^{n} f^i W_i$, $f^1, \ldots, f^n \in I_p^{(k)}(M, \mathscr{C})$, $W_1, \ldots, W_n \in \mathscr{W}$, such that

$$\sum_{i=1}^{n} (d_p^{(k)} f^i) \otimes W_i(p) = 0.$$

The modules $Z_p^{(k)}(\mathscr{W})$ and $Z_p^k(\mathscr{W})$ are closed with respect to localization. It is easy to see that if D^k is a complete differential of order k in a linear field module $((M, \mathscr{C}), \Phi, \mathscr{W})$, then $Z_p^k(\mathscr{W})$ contains those and only those fields $W \in Z_p^{(k-1)}(\mathscr{W})$ for which $D_p^k(W) = 0$. For an arbitrary open set $U \in \tau_{\mathscr{C}}$ the following equalities hold:

$$(3.3.1)\qquad (Z_p^{(k)}(\mathscr{W}))_U = Z_p^{(k)}(\mathscr{W}_U), \quad (Z_p^k(\mathscr{W}))_U = Z_p^k(\mathscr{W}_U).$$

The inclusion

$$(3.3.2)\qquad Z_p^{(k)}(\mathscr{W}) \subset Z_p^k(\mathscr{W}), \quad k \in N,$$

cannot, in general, be replaced by an equality.

EXAMPLE 3.3.1. Consider a differential space $(\boldsymbol{R}, C^\infty(\boldsymbol{R}))$, a positive integer r and an assignement Φ defined by the formula:

$$\Phi(p) = \begin{cases} \boldsymbol{R}^r, & p \neq 0, \\ \boldsymbol{R}^{r-1} \times \{0\}, & p = 0. \end{cases}$$

Let us include into the module $\mathscr{W}$ those and only those fields $(f^1, \ldots, f^r)$ for which $f^1, \ldots, f^{r-1} \in C^\infty(\boldsymbol{R})$ and $f^r \in I_0(\boldsymbol{R})$. Clearly,

$$Z_0^{(k)}(\mathscr{W}) = \{(f^1, \ldots, f^r); f^1, \ldots, f^{r-1} \in I_0^{k+1}, f^r \in I_0^{k+2}\}$$
$$\subsetneq \{(f^1, \ldots, f^r): f^1, \ldots, f^r \in I_0^{k+1}\} = Z_0^k(\mathscr{W}).$$

If the manifold M has a positive dimensions, then for any natural number k we have

$$Z_p^k(C^\infty(\xi)) \subsetneq Z_p^{(k-1)}(C^\infty(\xi)) = Z_p^{k-1}(C^\infty(\xi)).$$

In general, the equality on the right does not hold in pseudo-differential modules (Example 3.3.1) but it holds in differential modules.

THEOREM 3.3.1. *If a linear field module $((M, \mathscr{C}), \Phi, \mathscr{W})$ is a differential module, then $Z_p^{(k)}(\mathscr{W}) = Z_p^k(\mathscr{W})$, $k \in N$, $p \in M$.*

Proof. Every field $W \in Z_p^k(\mathscr{W})$ is of the form $\sum_{i=1}^n f^i \cdot W_i$ with functions $f^i \in I_p^{(k)}(M, \mathscr{C})$, $i = 1, \ldots, n$, satisfying the condition $\sum_{i=1}^n d_p^{(k)} f^i \otimes W_i(p) = 0$. There exist a neighbourhood U of p and fields $V_1, \ldots, V_r \in \mathscr{W}$ such that the fields $V_1|U, \ldots, V_r|U$ are a vector basis for the module $\mathscr{W}_U$ and $W_i|U = (\sum_{j=1}^r \lambda_i^j V_j)|U$, $i = 1, \ldots, n$, for certain functions $\lambda_i^j \in \mathscr{C}$. Thus by Lemma 2.1

$$0 = \sum_{i=1}^n d_p^{(k)} f^i \otimes W_i(p) = \sum_{i=1}^n d_p^{(k)} f^i \otimes \sum_{j=1}^r \lambda_i^j(p) \cdot V_j(p)$$
$$= \sum_{j=1}^r \Big(\sum_{i=1}^n (d_p^{(k)} f^i) \lambda_i^j(p)\Big) \otimes V_j(p) = \sum_{j=1}^r d_p^{(k)} \Big(\sum_{i=1}^n f^i \lambda_i^j\Big) \otimes V_j(p).$$

From the fact that the vectors $V_1(p), \ldots, V_r(p)$ are linearly independent we obtain the equalities $d_p^{(k)}(\sum_{i=1}^n f^i \lambda_i^j) = 0$, $j = 1, \ldots, r$, and from Theorem 2.1

we get the relation $\Psi^j = \sum_{i=1}^{n} f^i \lambda_i^j \in I_p^{(k+1)}(M, \mathscr{C}), j = 1, \ldots, r$. Thus $W|U = (\sum_{i=1}^{n} f^i W_i)|U = (\sum_{j=1}^{r} \Psi^j V_j)|U$, which means that $W \in Z_p^{(k)}(\mathscr{W})$. q.e.d.

THEOREM 3.3.2. *If a differential space $(M, \mathscr{C})$ is of class $\mathscr{D}_0$ and if we have $(M, \mathscr{C})_p = (M, \mathscr{C})'_p$ at a point $p \in M$, then $Z_p^{(k)}(\mathscr{X}(M, \mathscr{C})) = Z_p^k(\mathscr{X}(M, \mathscr{C}))$, $k \in N$. Consequently the set of points $p \in M$ for which the two modules are equal is dense in $\tau_{\mathscr{C}}$ and covers the set M'.*

Proof. From Theorem 1.1.1 follows the existence of a neighbourhood U of p such that, for any $q \in U$, $\dim(M, \mathscr{C})_q = \dim(M, \mathscr{C})_p$ and $(M, \mathscr{C})'_q = (M, \mathscr{C})_q$. Therefore the module $\mathscr{X}(U, \mathscr{C}_U)$ is differential. From the preceding theorem follows the equality

$$Z_p^{(k)}(\mathscr{X}(U, \mathscr{C}_U)) = Z_p^k(\mathscr{X}(U, \mathscr{C}_U)).$$

It is easy to prove the present theorem applying equalities (3.3.1). q.e.d.

3.4. The mapping $d_p^{(k)}$ for linear field modules. Condition $*$k). In ([6]) R. Palais has proved the existence and uniqueness of an R-linear mapping d_p^k: $Z_p^{k-1}(C^\infty(\xi)) \to L_s^k(M_p, \xi_p)$, $k \geqslant 1$, such that if $W \in Z_p^{k-1}(C^\infty(\xi))$ and $h \in C^\infty(\xi^*)$, then

$$d_p^k(h \circ W) = h(p) \circ d_p^k(W). \tag{3.4.1}$$

Note that $h \circ W \in I_p^k(M)$. If $W \in Z_p^{k-1}(C^\infty(\xi))$ and $W = \sum_{i=1}^{n} f^i W_i$, where $f^i \in I_p^k(M)$, $i = 1, \ldots, n$, then

$$d_p^k(W) = \sum_{i=1}^{n} d_p^k f^i \otimes W_i(p). \tag{3.4.2}$$

Indeed, let us consider a field $h \in C^\infty(\xi^*)$ and vectors $v_1, \ldots, v_k \in M_p$. From Lemma 2.1 follows

$$\begin{aligned} d_p^k(h \circ W)(v_1, \ldots, v_k) &= \sum_{i=1}^{n} d_p^k(f^i(h \circ W_i))(v_1, \ldots, v_k) \\ &= \sum_{i=1}^{n} (d_p^k f^i)(h \circ W_i)(p)(v_1, \ldots, v_k) \\ &= \sum_{i=1}^{n} (d_p^k f^i)(v_1, \ldots, v_k)(h \circ W_i)(p) \\ &= h(p)\Big(\sum_{i=1}^{n} (d_p^k f^i)(v_1, \ldots, v_k)\, W_i(p)\Big) \\ &= h(p)\Big(\Big(\sum_{i=1}^{n} d_p^k f^i \otimes W_i(p)\Big)(v_1, \ldots, v_k)\Big). \end{aligned}$$

Applying the formula analogous to (3.4.2) we define the mapping $d_p^{(k)}$ for linear field modules. Let $((M, \mathscr{C}), \Phi, \mathscr{W})$ be a linear field module.

DEFINITION 3.4.1. We denote by $d_p^{(k)}$, $p \in M$, $k \in N$, an R-linear mapping

$$d_p^{(k)}\colon Z_p^{(k-1)}(\mathscr{W}) \to L_s^k((M, \mathscr{C})_p', \Phi_{\mathscr{W}}(p))$$

such that $d_p^{(k)}(f \cdot W) = d_p^{(k)} f \otimes W(p)$ for $f \in I_p^{(k)}(M, \mathscr{C})$, $W \in \mathscr{W}$.

THEOREM 3.4.1. *A mapping $d_p^{(k)}$ exists if and only if the following condition is satisfied*:

$*$k) *if* $\sum_{i=1}^{n} f^i W_i = 0$, *where* $f^i \in I_p^{(k)}(M, \mathscr{C})$, $W_i \in \mathscr{W}$, $i = 1, \ldots, n$, $n \in N$, *then* $\sum_{i=1}^{n} d_p^{(k)} f^i \otimes W_i(p) = 0$.

There exists at most one mapping $d_p^{(k)}$.

Proof. If $d_p^{(k)}$ exists and if $\sum_{i=1}^{n} f^i W_i = 0$ for $f^i \in I_p^{(k)}(M, \mathscr{C})$, $W_i \in \mathscr{W}$, then $\sum_{i=1}^{n} d_p^{(k)} f^i \otimes W_i(p) = \sum_{i=1}^{n} d_p^{(k)}(f^i W_i) = d_p^{(k)}(\sum_{i=1}^{n} f^i W_i) = 0$, so that condition $*$k) is satisfied. The existence and uniqueness of the mapping $d_p^{(k)}$ under condition $*$k) is a consequence of the property that any field $W \in Z_p^{(k-1)}(\mathscr{W})$ is of the form $\sum_{i=1}^{n} f^i W_i$, $f^i \in I_p^{(k)}(M, \mathscr{C})$, and that $d_p^{(k)}(W) = \sum_{i=1}^{n} d_p^{(k)} f^i \otimes W_i(p)$ does not depend on the representation of the field W in this form. Thus the last formula defines the desired R-linear mapping. q.e.d.

It follows directly from the definition of a complete differential of order k that if a complete differential exists in a linear field module, then condition $*$k) is fulfilled at every point of the underlying space. Condition $*$k) need not be satisfied in every linear field module.

EXAMPLE 3.4.1. Consider a differential space $(\boldsymbol{R}, C^\infty(\boldsymbol{R}))$ and the assignement Φ defined as follows: $\Phi(p) = 0$ for $p \neq 0$ and $\Phi(0) = R$. Let $\mathscr{W}$ be the module of the all linear Φ-fields. For an arbitrary function $f \in I_0^k \setminus I_0^{k+1}$ and the field $W \in \mathscr{W}$ equal 1 at the point 0 we have

$$f \cdot W = 0 \quad \text{and} \quad d_0^{(k)} f \otimes W(p) \neq 0.$$

Remark. Let $((M, \mathscr{C}), \Phi, \mathscr{W})$ be a linear field module. For arbitrary fields $W \in Z_p^{(k-1)}(\mathscr{W})$ and $h \in \mathscr{W}^*$ we have

$$h \circ W \in I_p^{(k)}(M, \mathscr{C}) \quad \text{and} \quad d_p^{(k)}(h \circ W) = h(p) \circ d_p^{(k)}(W).$$

Proof. Assume that the field W is of the form

$$\sum_{i=1}^{n} f^i W_i \quad \text{for } f^i \in I_p^{(k)}(M, \mathscr{C}),\ i = 1, \ldots, n.$$

For any vectors $v_1, \ldots, v_k \in (M, \mathscr{C})'_p$

$$d_p^{(k)}(h \circ W)(v_1, \ldots, v_k) = h(p) \circ \Big(\sum_{i=1}^{n} d_p^{(k)} f^i \otimes W_i(p)\Big)(v_1, \ldots, v_k)$$

$$= h(p)\Big(d_p^{(k)}\Big(\sum_{i=1}^{n} f^i W_i\Big)\Big)(v_1, \ldots, v_k)$$

$$= h(p) \circ d_p^{(k)}(W)(v_1, \ldots, v_k). \quad \text{q.e.d.}$$

Condition $*$k) is satisfied in a fairly broad class of linear field modules (see Theorem 1.4.1).

THEOREM 3.4.2. *Let* $((M, \mathscr{C}), \Phi, \mathscr{W})$ *be a linear field module. If this module satisfies at a point* p *the conditions*:

(a) $\dim \Phi_{\mathscr{W}}(p) < \infty$,

(b) $\Psi_{\mathscr{W}^*}(p) = (\Phi_{\mathscr{W}}(p))^*$, *where* $\Psi(q) = (\Phi_{\mathscr{W}}(q))^*$, $q \in M$,

then for $k \in N$

(A) *there exists exactly one* R*-linear mapping*

$$d_p^{[k]}\colon Z_p^{(k-1)}(\mathscr{W}) \to L_s^k((M, \mathscr{C})'_p, \Phi_{\mathscr{W}}(p))$$

satisfying the equality

(3.4.3) $\quad d_p^{(k)}(h \circ W) = h(p) \circ d_p^{[k]}(W) \quad$ *for* $W \in Z_p^{(k-1)}(\mathscr{W})$ *and* $h \in \mathscr{W}^*$;

(B) *the module satisfies condition* $*$k) *at the point* p *and* $d_p^{[k]} = d_p^{(k)}$.

Proof. Assume that conditions (a) and (b) are satisfied at a point $p \in M$. For an arbitrarily fixed field $W \in Z_p^{(k-1)}(\mathscr{W})$ there exists the R-linear mapping $\Psi_{\mathscr{W}^*}(p) \ni w \mapsto d_p^{(k)}(h \circ W)$, where $h \in \mathscr{W}^*$ and $h(p) = w$; and for any collection of vectors $v_1, \ldots, v_k$ from $(M, \mathscr{C})'_p$ there exists exactly one element $d_p^{[k]}(W)(v_1, \ldots, v_k) \in \Phi_{\mathscr{W}}(p)$ such that

$$d_p^{(k)}(h \circ W)(v_1, \ldots, v_k) = h(p)\big(d_p^{[k]}(W)(v_1, \ldots, v_k)\big)$$

for $h \in \mathscr{W}^*$. The mapping

$$d_p^{[k]}(W) = \Big(\bigtimes_k (M, \mathscr{C})'_p \ni (v_1, \ldots, v_k) \mapsto d_p^{[k]}(W)(v_1, \ldots, v_k) \in \Phi_{\mathscr{W}}(p)\Big)$$

is symmetric and k-linear; it defines a linear mapping

$$d_p^{[k]}\colon Z_p^{(k-1)}(\mathscr{W}) \to L_s^k((M, \mathscr{C})'_p, \Phi_{\mathscr{W}}(p)).$$

This is the only mapping which has property (3.4.3) and we have $d_p^{(k)} = d_p^{[k]}$.

Now we show that condition $*$k) is fulfilled at the point $p \in M$. Let us consider any functions $f^1, \ldots, f^n \in I_p^{(k)}(M, \mathscr{C})$ and fields $W_1, \ldots, W_n \in \mathscr{W}$ such that $\sum_{i=1}^{n} f^i W_i = 0$. For any field $h \in \mathscr{W}^*$ and vectors $v_1, \ldots, v_k \in (M, \mathscr{C})'_p$

$$0 = h(p)\Big(d_p^{(k)}\big(\sum_{i=1}^{n} f^i W_i\big)(v_1, \ldots, v_k)\Big)$$

$$= h(p)\big(\sum_{i=1}^{n} (d_p^{(k)} f^i) \otimes W_i(p)\big)(v_1, \ldots, v_k).$$

From assumption (a) and (b) follows

$$\sum_{i=1}^{n} (d_p^{(k)} f^i) \otimes W_i(p) = 0. \quad \text{q.e.d.}$$

THEOREM 3.4.3. *If a linear field module* $((M, \mathscr{C}), \Phi, \mathscr{W})$ *satisfies at* $p \in M$ *the following conditions*:

(a) $\dim(M, \mathscr{C})'_p < \infty$,

(b) $\dim \Phi_{\mathscr{W}}(p) < \infty$,

(c) $*k)$,

then the following sequence is exact:

$$0 \to Z_p^k(\mathscr{W}) \hookrightarrow Z_p^{(k-1)}(\mathscr{W}) \xrightarrow{d_p^{(k)}} L_s^k((M, \mathscr{C})'_p, \Phi_{\mathscr{W}}(p)) \to 0. \tag{3.4.4}$$

Proof. It suffices to show the surjectivity of the mapping $d_p^{(k)}$ in the case when $\dim \Phi_{\mathscr{W}}(p) > 0$. Let us take an arbitrary element $\tau \in L_s^k((M, \mathscr{C})'_p, \Phi_{\mathscr{W}}(p))$ and a basis $v_1, \ldots, v_k$ of the space $\Phi_{\mathscr{W}}(p)$. There exist elements $\tau^1, \ldots, \tau^r \in L_s^k((M, \mathscr{C})'_p, R)$ such that $\tau = \sum_{i=1}^{r} \tau^i \otimes v_i$. From Theorem 2.1 we conclude that there exist functions $f^1, \ldots, f^r \in I_p^{(k)}(M, \mathscr{C})$ such that $d_p^{(k)} f^i = \tau^i$, $i = 1, \ldots, r$. For any fields $W_1, \ldots, W_r \in \mathscr{W}$ such that $W_i(p) = v_i$, $i = 1, \ldots, r$, the equality $d_p^{(k)}\big(\sum_{i=1}^{n} f^i W_i\big) = \tau$ is satisfied. q.e.d.

In what follows we assume that all linear field modules under consideration satisfy the assumptions of the last theorem.

From the definition of the mapping $d_p^{(k)}$ follows the equality: $Z_p^k(\mathscr{W}) = \ker d_p^{(k)}$. Therefore there exists a linear isomorphism

$$\varrho_p^k\colon Z_p^{(k-1)}(\mathscr{W})/Z_p^k(\mathscr{W}) \to L_s^k((M, C)'_p, \Phi_{\mathscr{W}}(p))$$

with the property $\varrho_p^k(W + Z_p^k(\mathscr{W})) = d_p^{(k)}(W)$ for $W \in Z_p^{(k-1)}(\mathscr{W})$. The inverse isomorphism will be denoted by i_p^k; it will be considered as an injective linear mapping

$$i_p^k\colon L_s^k((M, \mathscr{C})'_p, \Phi_{\mathscr{W}}(p)) \to \mathscr{W}/Z_p^k(\mathscr{W}).$$

Since $Z_p^k(\mathscr{W}) \subset Z_p^{(k-1)}(\mathscr{W})$, there exists the canonical surjective linear mapping

$$r_p^{k,(k-1)}\colon \mathscr{W}/Z_p^k(\mathscr{W}) \to \mathscr{W}/Z_p^{(k-1)}(\mathscr{W})$$

with the kernel $Z_p^{(k-1)}(\mathscr{W})/Z_p^k(\mathscr{W})$ (equal to im i_p^k). Hence the following sequence is exact:

$$0 \to L_s^k\big((M, \mathscr{C})'_p, \Phi_{\mathscr{W}}(p)\big) \xrightarrow{i_p^k} \mathscr{W}/Z_p^k(\mathscr{W}) \xrightarrow{r_p^{k,(k-1)}} \mathscr{W}/Z_p^{(k-1)}(\mathscr{W}) \to 0. \tag{3.4.5}$$

DEFINITION 3.4.2. Consider an arbitrary non-negative number k, a linear field module $\big((M, \mathscr{C}), \Phi, \mathscr{W}\big)$ and a point $p \in M$. Denote by

$$j_p^k\colon \mathscr{W} \to \mathscr{W}/Z_p^k(\mathscr{W}) \quad \text{and} \quad j_p^{(k)}\colon \mathscr{W} \to \mathscr{W}/Z_p^{(k)}(\mathscr{W})$$

the canonical linear mappings. The spaces

$$J_p^k(\mathscr{W}) = \mathscr{W}/Z_p^k(\mathscr{W}) \quad \text{and} \quad J_p^{(k)}(\mathscr{W}) = \mathscr{W}/Z_p^{(k)}(\mathscr{W})$$

will be called the *jet spaces*, of order k and (k), respectively, at the point p.

For any field $W \in Z_p^{(k-1)}(\mathscr{W})$

$$j_p^k(W) = i_p^k\big(d_p^{(k)}(W)\big). \tag{3.4.6}$$

Indeed, $i_p^k\big(d_p^{(k)}(W)\big) = i_p^k\big(\varrho_p^k(W + Z_p^k(\mathscr{W}))\big) = i_p^k\big(\varrho_p^k(j_p^k(W))\big) = j_p^k(W)$.

LEMMA 3.4.1. *If D^k is a complete differential of order k in a linear field module $\big((M, \mathscr{C}), \Phi, \mathscr{W}\big)$, then for any point $p \in M$ there exists exactly one R-linear mapping*

$$T_p\colon J_p^k(\mathscr{W}) \to L_s^k\big((M, \mathscr{C})'_p, \Phi_{\mathscr{W}}(p)\big)$$

such that $D_p^k = T_p \circ j_p^k$. It will be called the mapping linearizing the complete differential D^k at the point p. It satisfies the condition: $T_p \circ i_p^k = \mathrm{id}$.

Proof. If there exists a mapping linearizing the complete differential D^k at a point p, then it is defined by the formula

$$T_p\big(j_p^k(W)\big) = D_p^k(W); \tag{3.4.7}$$

therefore there exists at most one such mapping.

Consider the mapping T_p defined by formula (3.4.7). The formula defines the mapping T_p correctly because if $j_p^k(W) = j_p^k(W')$, then $(W - W') \in Z_p^k(\mathscr{W})$, which implies $D_p^k(W - W') = 0$. The equality $T_p \circ i_p^k = \mathrm{id}$ follows from $D_p^k(W) = d_p^{(k)}(W)$ for $W \in Z_p^{(k-1)}(\mathscr{W})$. q.e.d.

To conclude this subsection we prove one more important fact.

THEOREM 3.4.4. *If D^k is a complete differential of order k in a linear field module $\big((M, \mathscr{C}), \Phi, \mathscr{W}\big)$ and T_p is the mapping linearizing D^k at a point p, then*

$$\ker r_p^{k,(k-1)} \cap \ker T_p = 0.$$

Proof. Let us consider any field $W \in \mathscr{W}$ such that

$$j_p^k(W) \in \ker r_p^{k,(k-1)} \cap \ker T_p.$$

Then $W \in Z_p^{(k-1)}(\mathscr{W})$ and so $W = \sum_{i=1}^{n} f^i W_i$ for certain functions $f^1, \ldots, f^n \in I_p^{(k)}(M, \mathscr{C})$ and fields $W_1, \ldots, W_n \in \mathscr{W}$. Besides,

$$0 = T_p(j_p^k(W)) = D_p^k(W) = D_p^k\big(\sum_{i=1}^{n} f^i W_i\big) = \sum_{i=1}^{n} d_p^{(k)} f^i \otimes W_i(p);$$

hence $W \in Z_p^k(\mathscr{W})$ and consequently $j_p^k(W) = 0$. q.e.d.

3.5. Jet field module of order k and (k). An exact sequence of jet-modules.

DEFINITION 3.5.1. (a) The (k)-order jet field module, $k = 0, 1, \ldots$, of a linear field module $((M, \mathscr{C}), \Phi, \mathscr{W})$ is the least linear $(M \ni p \mapsto J_p^{(k)}(\mathscr{W}))$-field module closed with respect to localization, containing all fields of the form:

$$M \ni p \mapsto j_p^{(k)}(W) \in J_p^{(k)}(\mathscr{W}) \quad \text{for } W \in \mathscr{W}.$$

(b) The k-order jet field module, $k = 0, 1, 2, \ldots$, of a linear field module $((M, \mathscr{C}), \Phi, \mathscr{W})$ is the least $(M \ni p \mapsto J_p^k(\mathscr{W}))$-field module closed with respect to localization, containing all fields of the form:

(i) $$M \ni p \mapsto j_p^k(W) \in J_p^k(\mathscr{W}) \quad \text{for } W \in \mathscr{W},$$

(ii) $$M \ni p \mapsto i_p^k(S_p) \in J_p^k(\mathscr{W}) \quad \text{for } S \in L_s^k(\mathscr{X}(M, \mathscr{C}), \mathscr{W}).$$

It is clear that for any jet fields S of order k the field $(M \ni p \mapsto r_p^{k,(k-1)}(S_p))$ is a jet field of order $(k-1)$. Moreover, the mappings $i^k\colon L_s^k(\mathscr{X}(M, \mathscr{C}), \mathscr{W}) \to J^k(\mathscr{W})$ and $r^{k,(k-1)}\colon J^k(\mathscr{W}) \to J^{(k-1)}(\mathscr{W})$ defined by the formula $i^k(L)(p) = i_p^k(L_p)$ for $L \in L_s^k(\mathscr{X}(M, \mathscr{C}), \mathscr{W})$ and $p \in M$, $r^{k,(k-1)}(L)(p) = r_p^{k,(k-1)}(L_p)$ for $L \in J^k(\mathscr{W})$ and $p \in M$, are homomorphisms of linear field modules. The following natural mappings are R-linear,

$$j^k\colon \mathscr{W} \to J^k(\mathscr{W}) \quad \text{and} \quad j^{(k)}\colon \mathscr{W} \to J^{(k)}(\mathscr{W}).$$

Notice also that $j^0\colon \mathscr{W} \to J^0(\mathscr{W})$ is a $\mathscr{C}$-linear mapping.

In the sequel of this section we shall examine the sequence

(3.5.1) $$0 \to L_s^k(\mathscr{X}(M, \mathscr{C}), \mathscr{W}) \xrightarrow{i^k} J^k(\mathscr{W}) \xrightarrow{r^{k,(k-1)}} J^{(k-1)}(\mathscr{W}) \to 0,$$

called the *jet-module sequence.*

THEOREM 3.5.1. *If a differential space $(M, \mathscr{C})$ is paracompact and C-normal, then the mapping $r^{k,(k-1)}$ in sequence* (3.5.1) *is a surjection.*

Proof. Let us consider an arbitrary field $W \in J^{(k-1)}(\mathscr{W})$. For any point $p \in M$ there exists a neighbourhood $U^p \in \tau_{\mathscr{C}}$ of p such that $W|U^p = \big(\sum_{i=1}^{n} f^i j^{(k-1)} W_i\big)|U^p$ for a certain positive integer n, functions $f^i \in \mathscr{C}$ and fields $W^i \in \mathscr{W}$, $i = 1, 2, \ldots, n$. According to paracompactness, we subordinate a locally finite family $(V_t, t \in T)$ to the family $(U^p, p \in M)$, and applying $\mathscr{C}$-normality we choose a smooth partition of unity $(\varphi_t)_{t\in T}$ subordinate to this covering. We define fields $\theta_t \in J^k(\mathscr{W})$, $t \in T$, by the formula $\theta_t = \sum_{i=1}^{n} f^i j^k(W_i)$ and we put $\theta = \sum_{t\in T} \varphi_t \theta_t$. Obviously $\theta \in J^k(\mathscr{W})$, and since $r_p^{k,(k-1)}(\theta_t(p)) = W(p)$ for $p \in V_t$, we have $r^{k,(k-1)}(\theta) = W$. q.e.d.

The exactness of sequence (3.5.1) at the term "$J^k(\mathscr{W})$" in the case $k = 1$ will be proved without additional assumptions about the module $\mathscr{W}$. In the general case it will be proved for a broad class of linear field modules containing pseudo-differential modules.

To show exactness let us consider an arbitrary field $S \in \ker r^{k,(k-1)}$ and notice that there exists exactly one field L such that $i_p^k(L_p) = S_p$ for any point $p \in M$. We shall check that $L \in \mathrm{L}_s^k(\mathscr{X}(M, \mathscr{C}), \mathscr{W})$. From the definition of the module $J^k(\mathscr{W})$ it follows that in a certain neighbourhood U of $p \in M$ the field S is of the form

$$S|U = i^k(Ł)|U + \Big(\sum_{j=1}^{n} f^j j^k(W_j)\Big)\Big|U$$

for some field $Ł \in \mathrm{L}_s^k(\mathscr{X}(M, \mathscr{C}), \mathscr{W})$, a positive integer n, functions $f^1, \ldots, f^n \in \mathscr{C}$ and fields $W_1, \ldots, W_n \in \mathscr{W}$. Since for any point $q \in U$,

$$0 = r_q^{k,(k-1)}(S_q) = r_q^{k,(k-1)}\Big(i_p^k(Ł_q) + \sum_{j=1}^{n} f^j(q) j_q^k(W_j)\Big)$$

$$= j_q^{(k-1)}\Big(\sum_{j=1}^{n} f^j(q)\, W_j\Big),$$

then $\sum_{j=1}^{n} f^j(q)\, W_j \in Z_q^{(k-1)}(\mathscr{W})$.

From equality (3.4.6) one can easily derive the equality

$$i_q^k(L_q) = S_q = i_q^k(Ł_q) + j_q^k\Big(\sum_{j=1}^{n} f^j(q)\, W_j\Big) = i_q^k(Ł_q) + i_q^k\Big(d_q^{(k)}\Big(\sum_{j=1}^{n} f^j(q)\, W_j\Big)\Big)$$

$$= i_q^k\Big(Ł_q + d_q^{(k)}\Big(\sum_{j=1}^{n} f^j(q)\, W_j\Big)\Big).$$

As i_q^k is an injection, we have

$$L_q = Ł_q + d_q^{(k)}\Big(\sum_{j=1}^{n} f^j(q)\, W_j\Big) \quad \text{for } q \in U. \tag{3.5.2}$$

To prove exactness it suffices to show that

$$\Big(U \ni q \mapsto d_q^{(k)}\big(\sum_{j=1}^{n} f^j(q)\, W_j\big)\big(V_1(q), \ldots, V_k(q)\big)\Big) \in \mathscr{W}_U$$

for $V_1, \ldots, V_k \in \mathscr{X}(U, \mathscr{C}_U)$.

THEOREM 3.5.2. *The sequence*

$$0 \to L(\mathscr{X}(M, \mathscr{C}), \mathscr{W}) \xrightarrow{i^1} J^1(\mathscr{W}) \xrightarrow{r^{1,0}} J^0(\mathscr{W})$$

is exact.

Proof. Since $\sum_{j=1}^{n} f^j(q)\, W_j \in Z_q^0(\mathscr{W})$ for $q \in U$, then in particular $(\sum_{j=1}^{n} f^j W_j)|U = 0$, and so $\sum_{j=1}^{n} (f^j - f^j(q))\, W_j \in Z_q^0(\mathscr{W})$. Hence

$$0 = d_q^{(1)}\big(\sum_{j=1}^{n} f^j W_j\big) = d_q^{(1)}\big(\sum_{j=1}^{n} (f^j - f^j(q))\, W_j\big) + d_q^{(1)}\big(\sum_{j=1}^{n} f^j(q)\, W_j\big),$$

and this produces the equalities:

$$\begin{aligned} d_q^{(1)}\big(\sum_{j=1}^{n} f^j(q)\, W_j\big)(V(q)) &= -d_q^{(1)}\big(\sum_{j=1}^{n} (f^j - f^j(q))\, W_j\big)(V(q)) \\ &= -\sum_{j=1}^{n} d_q^{(1)}(f^j - f^j(q)) \otimes W_j(q)(V(q)) \\ &= -\sum_{j=1}^{n} d_q^{(1)}(f^j - f^j(q))(V(q))\, W_j(q) \\ &= -\sum_{j=1}^{n} V(q)(f^j - f^j(q))\, W_j(q) \\ &= -\sum_{j=1}^{n} V(q)(f^j)\, W_j(q) = -\big(\sum_{j=1}^{n} V(f^j)\, W_j\big)(q). \end{aligned}$$

q.e.d.

THEOREM 3.5.3. *If a linear field module* $((M, \mathscr{C}), \Phi, \mathscr{W})$ *satisfies condition*:

whenewer W is a linear $\Phi_{\mathscr{W}}$-field such that, for any field $h \in \mathscr{W}^*$, the function $h \circ W$ is from the ring $\mathscr{C}$, then $W \in \mathscr{W}$,

then the sequence

$$0 \to L_s^k(\mathscr{X}(M, \mathscr{C}), \mathscr{W}) \xrightarrow{i^k} J^k(\mathscr{W}) \xrightarrow{r^{k,(k-1)}} J^{(k-1)}(\mathscr{W})$$

is exact.

Proof. We shall show that every point $p \in U$ has a neighbourhood $V \subset U$ such that

$$\Big(V \ni q \mapsto d_q^{(k)}\big(\sum_{j=1}^{n} f^j(q)\, W_j(V_1(q), \ldots, V_k(q))\big)\Big) \in \mathscr{W}_V = (\mathscr{W}_U)_V$$

for $V_1, \ldots, V_k \in \mathscr{X}(V, \mathscr{C}_V)$.

Let us consider a function $\gamma \in \mathscr{C}$ separating the point p in the set U, i.e. a function γ such that $\gamma | B_0 = 1$ for some neighbourhood B_0 of p and $\gamma | U_0 = 0$ for an open set U_0 such that $U_0 \cup U = M$. Obviously,

$$\gamma \cdot \sum_{j=1}^{n} f^j(q)\, W_j \in Z_q^{(k-1)}(\mathscr{W})$$

for any $q \in M$. We put $V = B_0$. Then for $q \in V$ we have

$$d_q^{(k)}\big(\gamma \cdot \sum_{j=1}^{n} f^j(q)\, W_j\big) = d_q^{(k)}\big(\sum_{j=1}^{n} f^j(q)\, W_j\big).$$

It suffices to show that

$$\Big(M \ni q \mapsto d_q^{(k)}\big(\gamma \cdot \sum_{j=1}^{n} f^j(q)\, W_j\big)\big(V_1(q), \ldots, V_k(q)\big)\Big) \in \mathscr{W}. \tag{3.5.3}$$

For an arbitrary field $h \in \mathscr{W}^*$ it follows from Theorem 3.4.2 that

$$\begin{aligned}
h(q)\Big(d_q^{(k)}\big(\gamma \cdot \sum_{j=1}^{n} f^j(q)\, W_j\big)\big(V_1(q), \ldots, V_k(q)\big)\Big) \\
&= d_q^{(k)}\Big(h \circ \big(\gamma \cdot \sum_{j=1}^{n} f^j(q)\, W_j\big)\big(V_1(q), \ldots, V_k(q)\big)\Big) \\
&= d_q^{(k)}\big(\sum_{j=1}^{n} \gamma \cdot f^j(q)\, h \circ W_j\big)\big(V_1(q), \ldots, V_k(q)\big) \\
&= \Big((V_1, \ldots, V_k)\big(\sum_{j=1}^{n} \gamma \cdot f^j(q)\, h \circ W_j\big)\Big)(q) \\
&= \sum_{j=1}^{n} f^j(q)\, [(V_1, \ldots, V_k)(\gamma \cdot h \circ W_j)](q) \\
&= \big(\sum_{j=1}^{n} f^j\, [(V_1, \ldots, V_k)(\gamma \cdot h \circ W_j)]\big)(q). \qquad \text{q.e.d.}
\end{aligned}$$

Now we present the announced example of a linear field module $((M, \mathscr{C}), \Phi, \mathscr{W})$ for which there exists a $\mathscr{C}$-linear mapping $L\colon \mathscr{X}(M, \mathscr{C}) \to \mathscr{W}$ which is not a linear Ψ-field for $\Psi = \big(M \ni q \mapsto L((M, \mathscr{C})'_p, \Phi_{\mathscr{W}}(p))\big)$.

EXAMPLE 3.5.1. Consider the differential space

$$(R, \mathscr{C}) = \Big(R, \big(S_C(\{\mathrm{id}_R, (R \ni x \mapsto |x|)\})\big)_R\Big).$$

This space is of class $\mathscr{D}_0$. Let $e_x \in (\boldsymbol{R}, C^\infty(\boldsymbol{R}))_x$ for $x \neq 0$ be unitary vector, i.e. such that $e_x(\mathrm{id}_R) = 1$. The tangent space $(R, \mathscr{C})_0$ is 2-dimensional, having as a basis the vectors e_0 and ω defined by the formulas:

$$e_0(\mathrm{id}_R) = 1, \quad e_0(|\cdot|) = 0; \quad \omega(\mathrm{id}_R) = 0, \quad \omega(|\cdot|) = 1.$$

The vector field $V = (R \ni x \mapsto x e_x \in (R, \mathscr{C})_x)$ is smooth because $V(\mathrm{id}_R) = \mathrm{id}_R$ and $V(|\cdot|) = |\cdot|$. It cannot be written in the form $\sum_{i=1}^{n} f^i W_i$ for any numbers $n \in N$, functions $f^1, \ldots, f^n \in I_0^{(1)}(R, \mathscr{C})$ and fields $W_1, \ldots, W_n \in \mathscr{X}(R, \mathscr{C})$. Every vector field $W \in \mathscr{X}(R, \mathscr{C})$ is equal 0 at the point 0 and so, if $V = \sum_{i=1}^{n} f^i W_i$ for functions f^i with $f^i(p) = 0$, $i = 1, \ldots, n$, then $V(\mathrm{id}_R) = \sum_{i=1}^{n} f^i W_i(\mathrm{id}_R)$. We thus would get the equality $\mathrm{id}_R = \sum_{i=1}^{n} f^i g_i$ for certain functions f^i, g_i, $i = 1, \ldots, n$, from the ideal $I_0(R, \mathscr{C})$, and this produces a contradiction:

$$1 = e_0(\mathrm{id}_R) = \sum_{i=1}^{n} e_0(f^i g_i) = 0.$$

Now take a jet field module of order 0 of the initial module and a $\mathscr{C}$-linear mapping $j^0\colon \mathscr{X}(R, \mathscr{C}) \to J^0(\mathscr{X}(M, \mathscr{C}))$. j^0 is not a linear Ψ-field because for the vector field V we have

$$V(0) = 0 \quad \text{and} \quad j^0(V)(0) = j_0^0(V) \neq 0.$$

A scalar product in a linear field module $((M, \mathscr{C}), \Phi, \mathscr{W})$ is a linear field $G \in L_s^2(W, \mathscr{C})$ such that $G(p)(v, v) > 0$ for $0 \neq v \in \Phi_{\mathscr{W}}(p)$ and $G^*\colon \mathscr{W} \to \mathscr{W}^*$ defined by the formula $G^*(V)(W) = G(V, W)$ is an isomorphism of linear field modules.

EXAMPLE 3.5.2. In the space $(R, \mathscr{C})$ from the preceding example every smooth vector field is of the form $f \cdot e$, where $f \in \mathscr{C}$ is a function such that $f(0) = 0$. Every linear field $h \in \mathscr{W}^*$ is of the form $f \cdot e^*$, where $f\colon R \to R$ is a function such that $f(0) = 0$, $f \cdot \mathrm{id}_R \in \mathscr{C}$ and $f \cdot |\cdot| \in \mathscr{C}$. The function f defined by the formula $f(x) = 1$ when $x \neq 0$ and $f(0) = 0$ can serve as example. We shall construct a scalar product G in the module $\mathscr{X}(R, \mathscr{C})$. We put $G(x)(e_x, e_x) = 1/x$ for $x \neq 0$ and, of course, $G(0) = 0$. As every function $f \in \mathscr{C}$ equal 0 at the point 0 is of the form $f(x) = x \cdot f_1(x) + |x| \cdot f_2(x)$, $x \in R$, where $f_1, f_2 \in \mathscr{C}$, we see that $G(V, W) \in \mathscr{C}$ for $V, W \in \mathscr{X}(R, \mathscr{C})$. Let us take the vector field $V = f \cdot \mathrm{id}_R \cdot e$ for any field $h \in \mathscr{W}^*$ of the form $f \cdot e^*$. Then $V \in \mathscr{X}(R, \mathscr{C})$

and $G(V, W) = h(W)$. It is clear that the form $G(x)$, $x \in R$, is positive, and so G is a scalar product.

In the module $\mathscr{X}(R, \mathscr{C})$ there exists a symmetric covariant derivative determined by the scalar product just constructed. Notice that if $f, g \in \mathscr{C}$ and $f(0) = g(0) = 0$, then the function g is differentiable except at zero and the function h defined by the formula

$$h(x) = f(x)g'(x) \quad \text{for } x \neq 0 \text{ and } h(0) = 0$$

is from the ring $\mathscr{C}$. It is easy to prove that the following formula defines the generated covariant derivative:

$$(\nabla_{f\cdot e}\, g\cdot e)(x) = \begin{cases} \big(f\cdot g' - g\cdot f/(2\cdot \mathrm{id}_R)\big)(x)\, e_x, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

4. Complete differentials of higher order in relation to splittings of a sequence of jet-modules.

DEFINITION 4.1. A *splitting of the exact sequence of jet-modules*

$$0 \to L^k_s(\mathscr{X}(M, \mathscr{C}), \mathscr{W}) \to J^k(\mathscr{W}) \to J^{(k-1)}(\mathscr{W}) \to 0$$

(also a *connection* in the case $k = 1$) is an assignment

$$M \ni p \mapsto \mathscr{T}_p \subset J^k_p(\mathscr{W})$$

satisfying the conditions:

(i) $\mathscr{T}_p$ is a linear subspace of the space $J^k_p(\mathscr{W})$,

(ii) $J^k_p(\mathscr{W}) = \mathscr{T}_p \oplus \ker r^{k,(k-1)}_p$, $p \in M$,

(iii) if $P_p\colon J^k_p(\mathscr{W}) \to \ker r^{k,(k-1)}_p$ is the projection defined by the above direct sum then for $S \in J^k(\mathscr{W})$ the field

$$P(S) = \big(M \ni p \mapsto P_p(S_p)\big)$$

belongs to the module $J^k(\mathscr{W})$.

THEOREM 4.1. *If D^k is a complete differential of order k in a linear field module $\big((M, \mathscr{C}), \Phi, \mathscr{W}\big)$ and T_p is a mapping linearizing this differential at a point $p \in M$, then the assignment $M \ni p \mapsto \ker T_p$ is a splitting of the exact jet-module sequence of order k.*

Proof. Theorem 3.4.4 states that $\ker T_p \cap \ker r^{k,(k-1)}_p = 0$. For any element $j^k_p(W) \in J^k_p(\mathscr{W})$, $W \in \mathscr{W}$, we have $j^k_p(W) = i^k_p(D^k_p(W)) + \big(i^k_p(-D^k_p(W)) + j^k_p(W)\big)$ and $i^k_p(D^k_p(W)) \in \ker r^{k,(k-1)}_p$ and, by Lemma 3.4.1, $T_p\big(i^k_p(-D^k_p(W)) + j^k_p(W)\big) = -D^k_p(W) + D^k_p(W) = 0$; thus condition (ii) is fulfilled. Now consider an arbitrary field $S \in J^k(\mathscr{W})$; in a certain neighbourhood U of p the field S is of the form $S|U = i^k(L)|U + \big(\sum_{j=1}^{n} f^j j^k(W_j)\big)|U$ for a certain field

$Ł \in L^k_s(\mathscr{X}(M, \mathscr{C}), \mathscr{W})$, a number $n \in N$, functions $f^1, \ldots, f^n \in \mathscr{C}$ and fields $W_1, \ldots, W_n \in \mathscr{W}$. Hence

$$\begin{aligned} P(S)|U &= P\big(i^k(Ł) + \sum_{j=1}^{n} f^j j^k(W_j)\big)|U \\ &= i^k(Ł)|U + \sum_{j=1}^{n} f^j|U \cdot P\big(j^k(W_j)\big)|U \\ &= i^k(Ł)|U + \sum_{j=1}^{n} f^j|U \cdot i^k\big(D^k(W_j)\big)|U. \\ &= \big(i^k(Ł) + \sum_{j=1}^{n} f^j \cdot i^k\big(D^k(W_j)\big)\big)|U \end{aligned}$$

is an element of the module $J^k(W)_U$. q.e.d.

THEOREM 4.2. *If $(\mathscr{T}_p)_{p \in M}$ is splitting of the exact jet-module sequence of order k, then there exists exactly one homomorphism of linear field modules*

$$T\colon J^k(\mathscr{W}) \to L^k_s(\mathscr{X}(M, \mathscr{C}), \mathscr{W})$$

such that:

(i) $\ker T_p = \mathscr{T}_p$, $p \in M$,

(ii) $T \circ i^k = \mathrm{id}$.

Moreover, $T \circ j^k$ is a complete differential of order k in the module $\mathscr{W}$.

Proof. Consider the projection P_p and the projection $R_p\colon J^k_p(\mathscr{W}) \to \mathscr{T}_p$ defined by the direct sum $J^k_p(\mathscr{W}) = \ker r^{k,(k-1)}_p \oplus \mathscr{T}_p$, $p \in M$. Since $P_p\big(j^k_p(W)\big) \in \ker r^{k,(k-1)}_p = \operatorname{im} i^k_p$ for $W \in \mathscr{W}$, there exists exactly one element $s_p \in L^k_s\big((M, \mathscr{C})'_p, \Phi_{\mathscr{W}}(p)\big)$ associated with $W \in \mathscr{W}$ such that $P_p\big(j^k_p(W)\big) = i^k_p(s_p)$. Hence for $W \in \mathscr{W}$

$$T_p\big(j^k_p(W)\big) = T_p\big(R_p\big(j^k_p(W)\big) + P_p\big(j^k_p(W)\big)\big) = T_p\big(P_p\big(j^k_p(W)\big)\big) = T_p\big(i^k_p(s_p)\big) = s_p.$$

This proves the uniqueness of the mapping T_p and gives the method of computing it. Now it must be proved that

$$T(S) = \big(M \ni p \mapsto T_p(S_p)\big) \in L^k_s(\mathscr{X}(M, \mathscr{C}), \mathscr{W})$$

for any field $S \in J^k(\mathscr{W})$. As in the foregoing theorem, S will be given in the form $S|U = i^k(Ł)|U + \big(\sum_{j=1}^{n} f^j j^k(W_j)\big)|U$. Then

$$T(S)|U = T\big(i^k(Ł) + \sum_{j=1}^{n} f^j j^k(W_j)\big)|U = Ł|U + \sum_{j=1}^{n} \big(f^j \cdot T\big(j^k(W_j)\big)\big)|U.$$

From the exactness of the jet-module sequence of order k follows the existence of fields $Ł_j$, $j = 1, \ldots, n$, from the spaces $L^k_s(\mathscr{X}(M, \mathscr{C}), \mathscr{W})$ such that $P\big(j^k(W_j)\big) = i^k(Ł_j)$. Hence

$$T(S)|U = \big(Ł + \sum_{j=1}^{n} f^j Ł_j\big)|U.$$

It is easy to check that $T \circ j^k$ is an R-linear mapping. Finally, if $f \in \mathscr{C}$, $f - f(p) \in I_p^{(k)}(M, \mathscr{C})$ and $W \in \mathscr{W}$, then from (3.4.6) we have

$$\begin{aligned} T_p \circ j_p^k(f \cdot W) &= T_p \circ j_p^k\big((f - f(p))\, W\big) + T_p \circ j_p^k\big(f(p)\, W\big) \\ &= T_p\Big(i_p^k\Big(d_p^{(k)}\big((f - f(p))\, W\big)\Big)\Big) + f(p)\, T_p \circ j_p^k(W) \\ &= d_p^{(k)}\big(f - f(p)\big) \otimes W(p) + f(p)\, T_p \circ j_p^k(W). \qquad \text{q.e.d.} \end{aligned}$$

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